Lipschitz estimates for rough fractional multilinear integral operators on local generalized Morrey spaces

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Abstract

We obtain the Lipschitz boundedness for a class of fractional multilinear operators $I_{\Omega,\alpha}^{A,m}$ with rough kernels $\Omega \in L_s(\mathbb{S}^{n-1})$, $s > n/(n-\alpha)$ on the local generalized Morrey spaces $LM_{p,\varphi}^{\{x_0\}}$, generalized Morrey spaces $M_{p,\varphi}$ and vanishing generalized Morrey spaces $VM_{p,\varphi}$, where the functions A belong to homogeneous Lipschitz space $\dot{\Lambda}_\beta$, $0 < \beta < 1$. We find the sufficient conditions on the pair (φ_1, φ_2) which ensures the boundedness of the operators $I_{\Omega,\alpha}^{A,m}$ from $LM_{p,\varphi_1}^{\{x_0\}}$ to $LM_{q,\varphi_2}^{\{x_0\}}$, from M_{p,φ_1} to M_{q,φ_2} and from VM_{p,φ_1} to VM_{q,φ_2} for $1 < p < q < \infty$ and $1/p - 1/q = (\alpha+\beta)/n$. In all cases the conditions for the boundedness of the operator $I_{\Omega,\alpha}^{A,m}$ is given in terms of Zygmund-type integral inequalities on (φ_1, φ_2), which do not assume any assumption on monotonicity of $\varphi_1(x,r)$, $\varphi_2(x,r)$ in r.

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1 Introduction and results

It is well known that the fractional integrals and their commutators play an important role in harmonic analysis and PDE, where the definition of fractional integral operator with rough kernel I_α is

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n. \tag{1.1}$$

By the famous Hardy-Littlewood-Sobolev imbedding theorem (see [33]), we see that I_α maps $L_p(\mathbb{R}^n)$ continuously into $L_p(\mathbb{R}^n)$ with $1/p - 1/q = \alpha/n$ and $1 < p < n/\alpha$.

Let $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$, and γ_i $(i = 1, 2, \dots, n)$ be nonnegative integers. Denote $|\gamma| = \sum_{i=1}^n \gamma_i$ and

$$\gamma! = \gamma_1!\gamma_2! \dots \gamma_n!, \quad x^\gamma = x_1^{\gamma_1} x_2^{\gamma_2} \dots x_n^{\gamma_n},$$
$$D^\gamma = \frac{\partial^{|\gamma|}}{\partial^{\gamma_1} x_1 \, \partial^{\gamma_2} x_2 \dots \partial^{\gamma_n} x_n}.$$

Suppose that $\Omega \in L_s(\mathbb{S}^{n-1})$ $(s > 1)$ is homogeneous of degree zero on $\mathbb{R}^n$ with zero means value on $\mathbb{S}^{n-1}$, A is a function defined on $\mathbb{R}^n$. Following [5], the rough fractional multilinear integral operator $I_{\Omega,\alpha}^{A,m}$, is defined by

$$I_{\Omega,\alpha}^{A,m} f(x) = \int_{\mathbb{R}^n} \frac{R_m(A; x, y)}{|x-y|^{n-\alpha+m-1}} \Omega(x-y) f(y) dy, \tag{1.2}$$

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where $0 < \alpha < n$, and $R_m(A; x, y)$ is the m-th remainder of Taylor series of A at x about y. More precisely,

$$R_m(A; x, y) = A(x) - \sum_{|\gamma|<m} \frac{1}{\gamma!} D^\gamma A(y)(x-y)^\gamma. \tag{1.3}$$

Corresponding the rough fractional multilinear maximal operator $M_{\Omega,\alpha}^{A,m}$, is defined by

$$M_{\Omega,\alpha}^{A,m} f(x) = \sup_{r>0} \frac{1}{r^{n-\alpha+m-1}} \int_{B(x,r)} \frac{|R_m(A; x, y)|}{|x-y|^{n-\alpha+m-1}} |\Omega(x-y)| \, |f(y)| dy. \tag{1.4}$$

When $m = 1$, then $I_{\Omega,\alpha}^{A} \equiv I_{\Omega,\alpha}^{A,1}$ is just the commutator of the fractional integral $I_{\Omega,\alpha} f(x)$ with function A,

$$\begin{aligned} I_{\Omega,\alpha}^{A} f(x) &= \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} (A(x) - A(y)) f(y) dy \\ &= A(x) I_{\Omega,\alpha} f(x) - I_{\Omega,\alpha}(Af)(x) \equiv [A, I_{\Omega,\alpha}] f(x), \end{aligned}$$

where

$$I_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y) dy$$

and $M_{\Omega,\alpha}^{A} \equiv M_{\Omega,\alpha}^{A,1}$ is just the fractional maximal commutator of $M_{\Omega,\alpha}$ with function A,

$$M_{\Omega,\alpha}^{A} f(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{B(x,r)} \frac{|A(x) - A(y)|}{|x-y|^{n-\alpha}} |\Omega(x-y)| \, |f(y)| dy.$$

When $m \geq 2$, $I_{\Omega,\alpha}^{A,m}$ is a non-trivial generalization of the above commutator $[A, I_{\Omega,\alpha}]$.

Since the commutator has a close relation with partial differential equations and pseudo-differential operator, multilinear operator has been receiving more widely attention.

For $\beta > 0$, the homogeneous Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ is the space of functions f, such that

$$\|f\|_{\dot{\Lambda}_\beta} = \sup_{x,h \in \mathbb{R}^n, h \neq 0} \frac{\left|\Delta_h^{[\beta]+1} f(x)\right|}{|h|^\beta} < \infty,$$

where $\Delta_h^1 f(x) = f(x+h) - f(x)$, $\Delta_h^{k+1} f(x) = \Delta_h^k f(x+h) - \Delta_h^k f(x)$, $k \geq 1$. It is easy to see that if $\dot{\Lambda}_\beta(\mathbb{R}^n)$ and $0 < \beta < 1$, then for any $x, y \in \mathbb{R}^n$,

$$|f(x) - f(y)| \leq |x-y|^\beta \, \|f\|_{\dot{\Lambda}_\beta}.$$

Ding [5] proved the weighted (L_p, L_q)-boundedness of $I_{\Omega,\alpha}^{A,m}$, when $D^\gamma A \in L_r(\mathbb{R}^n)$, $1 < r \leq \infty$, $|\gamma| = m-1$. Wu and Yang [36] proved that if $D^\gamma A \in BMO(\mathbb{R}^n)$, $|\gamma| = m-1$, then $I_{\Omega,\alpha}^{A,m}$ is bounded on $L_p(\mathbb{R}^n)$.

On the other hand, in 1995, Paluszynski [24] studied the commutators generated by the Riesz potential and a Lipschitz function and gave some characterizations of the Besov spaces. Motivated by [24], it is natural to ask that what kinds of properties $I_{\Omega,\alpha}^{A,m}$ has when $D^{\gamma}A$ belongs to the Lipschitz class. In [19] the authors was prove that if $1 \le p < q < \infty$ and $1/p - 1/q = (\alpha+\beta)/n$, then the operator $I_{\Omega,\alpha}^{A,m}$ is bounded from $L_p(\mathbb{R}^n)$ into $L_q(\mathbb{R}^n)$ for $p > 1$ and from $L_p(\mathbb{R}^n)$ into $WL_q(\mathbb{R}^n)$ for $p \ge 1$.

In [19], Lu and Zhang proved the following result.

Theorem A. *Let $0 < \alpha < n$, $0 < \beta < 1$, $\alpha + \beta < n$, $1 < p < n/(\alpha+\beta)$, $1/p - 1/q = (\alpha+\beta)/n$ and $\Omega \in L_s(\mathbb{S}^{n-1})$, $s > n/(n-\alpha-\beta)$ is homogeneous of degree zero on $\mathbb{R}^n$. Assume that A has derivatives of order $m-1$ in $\dot{\Lambda}_{\beta}(\mathbb{R}^n)$. Then there exists a constant C, independent of A and f, such that*

$$\|M_{\Omega,\alpha}^{A,m} f\|_{L_q(\mathbb{R}^n)} \lesssim \|I_{\Omega,\alpha}^{A,m} f\|_{L_q(\mathbb{R}^n)} \lesssim \sum_{|\gamma|=m-1} \|D^{\gamma}A\|_{\dot{\Lambda}_{\beta}} \|f\|_{L_p(\mathbb{R}^n)}$$

for $p > 1$ and

$$\|M_{\Omega,\alpha}^{A,m} f\|_{WL_q(\mathbb{R}^n)} \lesssim \|I_{\Omega,\alpha}^{A,m} f\|_{WL_q(\mathbb{R}^n)} \lesssim \sum_{|\gamma|=m-1} \|D^{\gamma}A\|_{\dot{\Lambda}_{\beta}} \|f\|_{L_p(\mathbb{R}^n)}$$

for $p \ge 1$. Here and in the sequel, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant C, independent of all important parameters, such that $A \le CB$. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

The classical Morrey spaces were originally introduced by Morrey in [21] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [3, 8, 9, 13, 21, 23, 26, 27, 28, 30]. The first author, Mizuhara and Nakai [11, 20, 22] introduced generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$ (see, also [12, 13, 31]). In [11, 13, 20, 22], the boundedness of the classical operators and their commutators in spaces $M_{p,\varphi}$ was also studied, see also [2, 6, 14, 32].

For brevity, in the sequel we use the notations

$$\mathfrak{A}_{p,\varphi}(f;x,r) := r^{-n/p}\,\varphi(x,r)^{-1}\|f\|_{L_p(B(x,r))}$$

and

$$\mathfrak{A}_{\Phi,\varphi}^{W}(f;x,r) := r^{-n/p}\,\varphi(x,r)^{-1}\|f\|_{WL_p(B(x,r))}.$$

Definition 1.1. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. For any fixed $x_0 \in \mathbb{R}^n$ we denote by $LM_{p,\varphi}^{\{x_0\}} = LM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ the local generalized Morrey space, the space of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ with finite norm

$$\|f\|_{LM_{p,\varphi}^{\{x_0\}}} = \sup_{r>0} \mathfrak{A}_{p,\varphi}(f;x_0,r).$$

Also $WLM_{p,\varphi}^{\{x_0\}} = WLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ we denote the weak local generalized Morrey space, the space of all functions $f \in WL_p^{loc}(\mathbb{R}^n)$ with

$$\|f\|_{WLM_{p,\varphi}^{\{x_0\}}} = \sup_{r>0} \mathfrak{A}_{p,\varphi}^{W}(f;x_0,r) < \infty.$$

The local spaces $LM^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n)$ and $WLM^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n)$ are Banach spaces with respect to the norm

$$\|f\|_{LM^{\{x_0\}}_{p,\varphi}} = \sup_{r>0} \mathfrak{A}_{p,\varphi}(f;x_0,r), \quad \|f\|_{WLM^{\{x_0\}}_{p,\varphi}} = \sup_{r>0} \mathfrak{A}^W_{p,\varphi}(f;x_0,r),$$

respectively.

Remark 1.2. (i) When $\varphi(x,r) = r^{(\lambda-n)/p}$, $LM^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n)$ is the local (central) Morrey space $LM^{\{0\}}_{p,\lambda}(\mathbb{R}^n)$ studied in [1];

(ii) The local generalized Morrey space $LM^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n)$ were introduced by V.S. Guliyev in [11], see also [12, 15, 18] etc.

Definition 1.3. The vanishing generalized Morrey space $VM_{p,\varphi}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in M_{p,\varphi}(\mathbb{R}^n)$ such that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{p,\varphi}(f;x,r) = 0. \tag{1.5}$$

The vanishing weak generalized Morrey space $VWM_{p,\varphi}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in WM_{p,\varphi}(\mathbb{R}^n)$ such that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}^W_{p,\varphi}(f;x,r) = 0.$$

The vanishing spaces $VM_{p,\varphi}(\mathbb{R}^n)$ and $VWM_{p,\varphi}(\mathbb{R}^n)$ are Banach spaces with respect to the norm

$$\|f\|_{VM_{p,\varphi}} \equiv \|f\|_{M_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \mathfrak{A}_{p,\varphi}(f;x,r),$$
$$\|f\|_{VWM_{p,\varphi}} \equiv \|f\|_{WM_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \mathfrak{A}_{W,p,\varphi}(f;x,r),$$

respectively.

In the case $\varphi(x,r) = r^{(\lambda-n)/p}$ $VM_{p,\varphi}(\mathbb{R}^n)$ is the vanishing Morrey space $VM_{p,\lambda}$ introduced in [34], where applications to PDE were considered.

We refer to [17, 26, 29] for some properties of vanishing generalized Morrey spaces.

In [16], V.S. Guliyev proved the following result.

Theorem B. *Let $x_0 \in \mathbb{R}^n$ and $\Omega \in L_s(\mathbb{S}^{n-1})$, $1 < s \le \infty$ is homogeneous of degree zero on $\mathbb{R}^n$. Let also $0 < \alpha < n$, $1 \le p < n/\alpha$, $1/p - 1/q = \alpha/n$, $s' \le p$ or $q < s$, and $\varphi_1 \in \Omega_{p,\mathrm{loc}}$, $\varphi_2 \in \Omega_{q,\mathrm{loc}}$ satisfy the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,sup}_{t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{q}}} \frac{dt}{t} \le C\,\varphi_2(x_0,r), \tag{1.6}$$

where C does not depend on r. Then the operator $I_{\Omega,\alpha}$ is bounded from $LM^{\{x_0\}}_{p,\varphi_1}(\mathbb{R}^n)$ to $LM^{\{x_0\}}_{q,\varphi_2}(\mathbb{R}^n)$.

Corollary A. *Let $\Omega \in L_s(\mathbb{S}^{n-1})$, $1 < s \le \infty$ is homogeneous of degree zero on $\mathbb{R}^n$. Let also $0 < \alpha < n$, $1 \le p < n/\alpha$, $1/p - 1/q = \alpha/n$, $s' \le p$ or $q < s$, and $\varphi_1 \in \Omega_p$, $\varphi_2 \in \Omega_q$ satisfy the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,sup}_{t<\tau<\infty} \varphi_1(x,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{q}}} \frac{dt}{t} \le C\,\varphi_2(x,r), \tag{1.7}$$

where C does not depend on x and r. Then the operator $I_{\Omega,\alpha}$ is bounded from $M_{p,\varphi_1}(\mathbb{R}^n)$ to $M_{q,\varphi_2}(\mathbb{R}^n)$.

In this paper, we consider the boundedness of the fractional multilinear operators $I_{\Omega,\alpha}^{A,m}$ with rough kernels $\Omega \in L_s(\mathbb{S}^{n-1})$, $s > n/(n-\alpha)$ on the local generalized Morrey spaces $LM_{p,\varphi}^{\{x_0\}}$, generalized Morrey spaces $M_{p,\varphi}$ and vanishing generalized Morrey spaces $VM_{p,\varphi}$, where the functions A belong to homogeneous Lipschitz space $\dot{\Lambda}_\beta$, $0 < \beta < 1$.

Our main results can be formulated as follows.

Theorem 1.4. Let $x_0 \in \mathbb{R}^n$ and $\Omega \in L_s(\mathbb{S}^{n-1})$, $s > n/(n-\alpha-\beta)$ is homogeneous of degree zero on $\mathbb{R}^n$. Let also $0 < \alpha < n$, $0 < \beta < 1$, $\alpha+\beta < n$, $1 \le p < n/(\alpha+\beta)$ and $1/p - 1/q = (\alpha+\beta)/n$. Assume that A has derivatives of order $m-1$ in $\dot{\Lambda}_\beta(\mathbb{R}^n)$ and $\varphi_1 \in \Omega_{p,\mathrm{loc}}$, $\varphi_2 \in \Omega_{q,\mathrm{loc}}$ satisfy the condition (1.6) Then the operator $I_{\Omega,\alpha}^{A,m}$ is bounded from $LM_{p,\varphi_1}^{\{x_0\}}(\mathbb{R}^n)$ to $LM_{q,\varphi_2}^{\{x_0\}}(\mathbb{R}^n)$ for $p > 1$ and from $LM_{1,\varphi_1}^{\{x_0\}}(\mathbb{R}^n)$ to $WLM_{\frac{n}{n-\alpha-\beta},\varphi_2}^{\{x_0\}}(\mathbb{R}^n)$. Moreover, for $p > 1$

$$\|I_{\Omega,\alpha}^{A,m} f\|_{LM_{q,\varphi_2}^{\{x_0\}}} \lesssim \Big(\sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \Big) \|f\|_{LM_{p,\varphi_1}^{\{x_0\}}}$$

and for $p = 1$

$$\|I_{\Omega,\alpha}^{A,m} f\|_{WLM_{\frac{n}{n-\alpha-\beta},\varphi_2}^{\{x_0\}}} \lesssim \Big(\sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \Big) \|f\|_{LM_{1,\varphi_1}^{\{x_0\}}}.$$

Corollary 1.5. Let $\Omega \in L_s(\mathbb{S}^{n-1})$, $s > n/(n-\alpha-\beta)$ is homogeneous of degree zero on $\mathbb{R}^n$. Let also $0 < \alpha < n$, $0 < \beta < 1$, $\alpha+\beta < n$, $1 \le p < n/(\alpha+\beta)$ and $1/p - 1/q = (\alpha+\beta)/n$. Assume that A has derivatives of order $m-1$ in $\dot{\Lambda}_\beta(\mathbb{R}^n)$ and $\varphi_1 \in \Omega_p$, $\varphi_2 \in \Omega_q$ satisfy the condition (1.7) Then the operator $I_{\Omega,\alpha}^{A,m}$ is bounded from $M_{p,\varphi_1}(\mathbb{R}^n)$ to $M_{q,\varphi_2}(\mathbb{R}^n)$ for $p > 1$ and from $M_{1,\varphi_1}(\mathbb{R}^n)$ to $WM_{\frac{n}{n-\alpha-\beta},\varphi_2}(\mathbb{R}^n)$. Moreover, for $p > 1$

$$\|I_{\Omega,\alpha}^{A,m} f\|_{M_{q,\varphi_2}} \lesssim \Big(\sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \Big) \|f\|_{M_{p,\varphi_1}}$$

and for $p = 1$

$$\|I_{\Omega,\alpha}^{A,m} f\|_{WM_{\frac{n}{n-\alpha-\beta},\varphi_2}} \lesssim \Big(\sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \Big) \|f\|_{M_{1,\varphi_1}}.$$

Theorem 1.6. Let $\Omega \in L_s(\mathbb{S}^{n-1})$, $s > n/(n-\alpha-\beta)$ is homogeneous of degree zero on $\mathbb{R}^n$. Let also $0 < \alpha < n$, $0 < \beta < 1$, $\alpha+\beta < n$, $1 \le p < n/(\alpha+\beta)$ and $1/p - 1/q = (\alpha+\beta)/n$. Assume that A has derivatives of order $m-1$ in $\dot{\Lambda}_\beta(\mathbb{R}^n)$ and $\varphi_1 \in \Omega_{p,1}$, $\varphi_2 \in \Omega_{q,1}$ satisfies the conditions

$$c_\delta := \int_\delta^\infty \sup_{x\in\mathbb{R}^n} \varphi_1(x,t) \frac{dt}{t} < \infty$$

for every $\delta > 0$, and

$$\int_r^\infty \frac{\varphi_1(x,t)}{t^{1-\alpha-\beta}}\, dt \le C_0 \varphi_2(x,r), \tag{1.8}$$

where C_0 does not depend on $x \in \mathbb{R}^n$ and $r > 0$. Then the operator $I_{\Omega,\alpha}^{A,m}$ is bounded from VM_{p,φ_1} to VM_{q,φ_2} for $p > 1$ and from VM_{1,φ_1} to $WVM_{\frac{n}{n-\alpha-\beta},\varphi_2}$.

2 Some preliminaries

To prove the theorems, we need auxiliary results. The first one is the following characterizations of Lipschitz space, which is due to DeVore and Sharply [10].

Lemma 2.1. Let $0 < \beta < 1$. Then

$$\|f\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \approx \sup_B \frac{1}{|B|^{1+\beta/n}} \int_B |f(x) - f_B| dx.$$

Below we present some conclusions about $R_m(A; x, y)$.

Lemma 2.2. [25] Suppose A be a function on $\mathbb{R}^n$ with the m-th derivatives in $L_q^{\text{loc}}(\mathbb{R}^n)$, $q > n$. Then

$$|R_m(A; x, y)| \lesssim |x - y|^m \sum_{|\gamma|=m} \left(\frac{1}{B(x, 5\sqrt{n}|x-y|)} \int_{B(x,5\sqrt{n}|x-y|)} |D^\gamma A(z)| dz \right)^{1/q}.$$

We state the following important lemma.

Lemma 2.3. [35] Let $\Omega \in L_s(\mathbb{S}^{n-1})$, $s > n/(n - \alpha - \beta)$ is homogeneous of degree zero on $\mathbb{R}^n$. Let also $0 < \alpha < n$, $0 < \beta < 1$, $\alpha + \beta < n$ and $D^\gamma A \in \dot{\Lambda}_\beta(\mathbb{R}^n)$. Then

$$\left| I_{\Omega,\alpha}^{A,m} f(x) \right| \lesssim \Big(\sum_{|\gamma|=m-1} \left\| D^\gamma A \right\|_{\dot{\Lambda}_\beta} \Big) I_{|\Omega|,\alpha+\beta}(|f|)(x). \tag{2.1}$$

Finally, we present a relationship between essential supremum and essential infimum.

Lemma 2.4. [4, 37] Let f be a real-valued nonnegative function and measurable on E. Then

$$\Big(\operatorname*{ess\,inf}_{x \in E} f(x) \Big)^{-1} = \operatorname*{ess\,sup}_{x \in E} \frac{1}{f(x)}.$$

It is natural, first of all, to find conditions ensuring that the spaces $LM_{p,\varphi}^{\{x_0\}}$ and $M_{p,\varphi}$ are nontrivial, that is consist not only of functions equivalent to 0 on $\mathbb{R}^n$.

Lemma 2.5. Let $x_0 \in \mathbb{R}^n$, $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and $1 \le p < \infty$. If

$$\sup_{t<r<\infty} \frac{r^{-\frac{n}{p}}}{\varphi(x_0, r)} = \infty \quad \text{for some } t > 0, \tag{2.2}$$

then $LM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

Proof. Let (2.2) be satisfied and f be not equivalent to zero. Then $\|f\|_{L_p(B(x_0,t))} > 0$, hence

$$\begin{aligned}\|f\|_{LM^{\{x_0\}}_{p,\varphi}} &\geq \sup_{t<r<\infty} \varphi(x_0,r)^{-1} r^{-\frac{n}{p}} \|f\|_{L_p(B(x_0,r))} \\ &\geq \|f\|_{L_p(B(x_0,t))} \sup_{t<r<\infty} \varphi(x_0,r)^{-1} r^{-\frac{n}{p}}.\end{aligned}$$

Therefore $\|f\|_{LM^{\{x_0\}}_{p,\varphi}} = \infty$. Q.E.D.

Remark 2.6. We denote by $\Omega_{p,\mathrm{loc}}$ the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0,\infty)$ such that for all $t>0$,

$$\sup_{x\in\mathbb{R}^n} \Big\| \frac{r^{-\frac{n}{p}}}{\varphi(x,r)} \Big\|_{L_\infty(t,\infty)} < \infty.$$

In what follows, keeping in mind Lemma 2.5, for the non-triviality of the space $LM^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n)$ we always assume that $\varphi \in \Omega_{p,\mathrm{loc}}$.

Lemma 2.7. [7] Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \leq p < \infty$.

(i) If

$$\sup_{t<r<\infty} \frac{r^{-\frac{n}{p}}}{\varphi(x,r)} = \infty \quad \text{for some } t>0 \text{ and for all } x \in \mathbb{R}^n, \tag{2.3}$$

then $M_{p,\varphi}(\mathbb{R}^n) = \Theta$.

(ii) If

$$\sup_{0<r<\tau} \varphi(x,r)^{-1} = \infty \quad \text{for some } \tau>0 \text{ and for all } x \in \mathbb{R}^n, \tag{2.4}$$

then $M_{p,\varphi}(\mathbb{R}^n) = \Theta$.

Remark 2.8. We denote by Ω_p the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0,\infty)$ such that for all $t>0$,

$$\sup_{x\in\mathbb{R}^n} \Big\| \frac{r^{-\frac{n}{p}}}{\varphi(x,r)} \Big\|_{L_\infty(t,\infty)} < \infty, \quad \text{and} \quad \sup_{x\in\mathbb{R}^n} \Big\| \varphi(x,r)^{-1} \Big\|_{L_\infty(0,t)} < \infty,$$

respectively. In what follows, keeping in mind Lemma 2.7, for the non-triviality of the space $M_{p,\varphi}(\mathbb{R}^n)$ we always assume that $\varphi \in \Omega_p$.

Remark 2.9. We denote by $\Omega_{p,1}$ the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0,\infty)$ such that

$$\inf_{x\in\mathbb{R}^n} \inf_{r>\delta} \varphi(x,r) > 0, \text{ for some } \delta>0, \tag{2.5}$$

and

$$\lim_{r\to 0} \frac{r^{n/p}}{\varphi(x,r)} = 0,$$

For the non-triviality of the space $VM_{p,\varphi}(\mathbb{R}^n)$ we always assume that $\varphi \in \Omega_{p,1}$.

3 Guliyev type local estimates

In the following theorem we get Guliyev type local estimate (see, for example, [11, 13]) for the operator $I^{A,m}_{\Omega,\alpha}$.

Theorem 3.1. Let $x_0 \in \mathbb{R}^n$ and $\Omega \in L_s(\mathbb{S}^{n-1})$, $s > n/(n-\alpha-\beta)$ is homogeneous of degree zero on $\mathbb{R}^n$. Let also $0 < \alpha < n$, $0 < \beta < 1$, $\alpha+\beta < n$, $1 < p < n/(\alpha+\beta)$ and $1/p - 1/q = (\alpha+\beta)/n$. Assume that A has derivatives of order $m-1$ in $\dot{\Lambda}_\beta(\mathbb{R}^n)$, then the inequality

$$\|I^{A,m}_{\Omega,\alpha} f\|_{L_q(B(x_0,r))} \le C\Big(\sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta}\Big)\, r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))}\, t^{-\frac{n}{q}-1}\, dt \tag{3.1}$$

holds for any ball $B(x_0,r)$ and for all $f \in L_p^{\rm loc}(\mathbb{R}^n)$, $p > 1$. Moreover, for $p = 1$ the inequality

$$\|I^{A,m}_{\Omega,\alpha} f\|_{WL_q(B(x_0,r))} \le C\Big(\sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta}\Big)\, r^{n} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))}\, t^{-n-1}\, dt \tag{3.2}$$

holds for any ball $B(x_0,r)$ and for all $f \in L_1^{\rm loc}(\mathbb{R}^n)$, where the constant C independent of f, r and x_0.

Proof. We write f as $f = f_1 + f_2$, where $f_1(y) = f(y)_{\chi_{B(x_0,2r)}(y)}$, $\chi_{B(x_0,2r)}$ denotes the characteristic function of $B(x_0,2r)$. Then

$$\|I^{A,m}_{\Omega,\alpha} f\|_{L_q(B(x_0,r))} \le \|I^{A,m}_{\Omega,\alpha} f_1\|_{L_q(B(x_0,r))} + \|I^{A,m}_{\Omega,\alpha} f_2\|_{L_q(B(x_0,r))}.$$

Since $f_1 \in L_p(\mathbb{R}^n)$, by the boundedness of $T^A_{\Omega,\alpha}$ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ (Theorem A) we get

$$\begin{aligned}\|I^{A,m}_{\Omega,\alpha} f_1\|_{L_q(B(x_0,r))} &\le \|I^{A,m}_{\Omega,\alpha} f_1\|_{L_q(\mathbb{R}^n)} \\ &\lesssim \Big(\sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta}\Big) \|f_1\|_{L_p(\mathbb{R}^n)} \\ &= \Big(\sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta}\Big) \|f\|_{L_p(B(x_0,2r))}.\end{aligned}$$

Moreover, the following inequality

$$\begin{aligned}\|f\|_{L_p(B(x_0,2r))} &\lesssim r^{\frac{n}{q}}\, \|f\|_{L_p(B(x_0,2r))} \int_{2r}^{\infty} t^{-\frac{n}{q}-1} dt \\ &\le r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))}\, t^{-\frac{n}{q}-1} dt\end{aligned} \tag{3.3}$$

is valid. Thus

$$\|I^{A,m}_{\Omega,\alpha} f_1\|_{L_q(B(x_0,r))} \lesssim \Big(\sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta}\Big)\, r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))}\, t^{-\frac{n}{q}-1} dt. \tag{3.4}$$

By Lemma 2.3 we get

$$\begin{aligned}\left|I_{\Omega,\alpha}^{A,m} f_2(x)\right| &\lesssim \Big(\sum_{|\gamma|=m-1} \left\|D^\gamma A\right\|_{\dot\Lambda_\beta}\Big) I_{|\Omega|,\alpha+\beta}(|f_2|)(x)\\ &\lesssim \Big(\sum_{|\gamma|=m-1} \left\|D^\gamma A\right\|_{\dot\Lambda_\beta}\Big) \int_{\complement_{(B(x_0,2r))}} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha-\beta}}\, |f(y)|\, dy.\end{aligned}$$

To estimate $\|I_{\Omega,\alpha}^{A,m} f_2\|_{L_p(B(x_0,r))}$, obverse that $x \in B$, $y \in (2B)^c$ implies $|x-y| \approx |x_0-y|$. Then we have

$$\sup_{x\in B} |I_{\Omega,\alpha}^{A,m}(f_2)(x)| \lesssim \Big(\sum_{|\gamma|=m-1} \left\|D^\gamma A\right\|_{\dot\Lambda_\beta}\Big) \int_{\complement_{(2B)}} \frac{|\Omega(x-y)|\,|f(y)|}{|x_0-y|^{n-\alpha-\beta}}.$$

By Fubini's theorem we have

$$\begin{aligned}\int_{\complement_{(2B)}} \frac{|f(y)||\Omega(x-y)|}{|x_0-y|^{n-\alpha-\beta}} dy &\approx \int_{\complement_{(2B)}} |f(y)||\Omega(x-y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1-\alpha-\beta}} dy\\ &\approx \int_{2r}^{\infty} \int_{2r\le|x_0-y|\le t} |f(y)||\Omega(x-y)| dy \frac{dt}{t^{n+1-\alpha-\beta}}\\ &\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)||\Omega(x-y)| dy \frac{dt}{t^{n+1-\alpha-\beta}}.\end{aligned}$$

Applying Hölder's inequality, we get

$$\begin{aligned}&\int_{\complement_{(2B)}} \frac{|f(y)||\Omega(x-y)|}{|x_0-y|^{n-\alpha-\beta}} dy\\ &\lesssim \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))}\, \|\Omega(\cdot-y)\|_{L_s(B(x_0,r))}\, |B(x_0,t)|^{1-\frac{1}{p}-\frac{1}{s}}\, \frac{dt}{t^{n+1-\alpha-\beta}} \qquad (3.5)\\ &\lesssim \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))}\, t^{-\frac{n}{q}-1}\, dt.\end{aligned}$$

Moreover, for all $p \in [1,\infty)$ the inequality

$$\|I_{\Omega,\alpha}^{A,m} f_2\|_{L_q(B)} \lesssim \Big(\sum_{|\gamma|=m-1} \left\|D^\gamma A\right\|_{\dot\Lambda_\beta}\Big)\, r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))}\, t^{-\frac{n}{q}-1}\, dt. \qquad (3.6)$$

is valid. Thus, combining the estimates of (3.4) and (3.6), we have

$$\|I_{\Omega,\alpha}^{A,m} f\|_{L_q(B)} \lesssim \Big(\sum_{|\gamma|=m-1} \left\|D^\gamma A\right\|_{\dot\Lambda_\beta}\Big)\, r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))}\, t^{-\frac{n}{q}-1}\, dt.$$

Let $p=1<q<s\le\infty$. From the weak $(1,q)$ boundedness of $I_{\Omega,\alpha}$ and (3.3) it follows that:

$$\begin{aligned}\|I^{A,m}_{\Omega,\alpha}f_1\|_{WL_q(B)} &\le \|I^{A,m}_{\Omega,\alpha}f_1\|_{WL_q(\mathbb{R}^n)}\\ &\lesssim \Big(\sum_{|\gamma|=m-1}\Big\|D^\gamma A\Big\|_{\dot\Lambda_\beta}\Big)\|f_1\|_{L_1(\mathbb{R}^n)}\\ &= \Big(\sum_{|\gamma|=m-1}\Big\|D^\gamma A\Big\|_{\dot\Lambda_\beta}\Big)\|f\|_{L_1(2B)} \qquad (3.7)\\ &\lesssim \Big(\sum_{|\gamma|=m-1}\Big\|D^\gamma A\Big\|_{\dot\Lambda_\beta}\Big)r^{\frac{n}{q}}\int_{2r}^\infty \|f\|_{L_1(B(x_0,t))}\,t^{-\frac{n}{q}-1}\,dt.\end{aligned}$$

Then from (3.6) and (3.7) we get the inequality (3.2).
This completes the proof of Theorem 3.1. Q.E.D.

4 Proof of Theorem 1.4

Since $f\in LM^{\{x_0\}}_{p,\varphi_1}(\mathbb{R}^n)$, then by Lemma 2.4 and the non decreasing, respect to t, of the norm $\|f\|_{L_p(B(x_0,t))}$, we get

$$\begin{aligned}\frac{\|f\|_{L_p(B(x_0,t))}}{\operatorname*{ess\,inf}\limits_{t<\tau<\infty}\varphi_1(x_0,\tau)\tau^{\frac{n}{p}}} &\le \operatorname*{ess\,sup}_{t<\tau<\infty}\frac{\|f\|_{L_p(B(x_0,t))}}{\varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}\\ &\le \sup_{\tau>0}\frac{\|f\|_{L_p(B(x_0,\tau))}}{\varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}\le \|f\|_{LM^{\{x_0\}}_{p,\varphi_1}}.\end{aligned}$$

Since (φ_1,φ_2) satisfies (1.6), we have

$$\begin{aligned}&\int_r^\infty \|f\|_{L_p(B(x_0,t))}t^{-\frac{n}{q}-1}\,dt\\ &=\int_r^\infty \frac{\|f\|_{L_p(B(x_0,t))}}{\operatorname*{ess\,inf}\limits_{t<\tau<\infty}\varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}\,\frac{\operatorname*{ess\,inf}\limits_{t<\tau<\infty}\varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{q}}}\,\frac{dt}{t}\\ &\le \|f\|_{LM^{\{x_0\}}_{p,\varphi_1}}\int_r^\infty \frac{\operatorname*{ess\,inf}\limits_{t<\tau<\infty}\varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{q}}}\,\frac{dt}{t}\\ &\lesssim \varphi_2(x_0,t)\ \|f\|_{LM^{\{x_0\}}_{p,\varphi_1}}.\end{aligned}$$

Then by (3.1) we get

$$\begin{aligned}\|I_{\Omega,\alpha}^{A,m}f\|_{LM_{q,\varphi_2}^{\{x_0\}}} &= \sup_{t>0}\frac{1}{\varphi_2(x_0,t)}\Big(\frac{1}{|B(x_0,t)|}\int_{B(x_0,t)}|I_{\Omega,\alpha}^{A,m}f(y)|^q dy\Big)^{1/q}\\ &\lesssim \Big(\sum_{|\gamma|=m-1}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\Big)\sup_{t>0}\frac{1}{\varphi_2(x_0,t)}\int_r^\infty \|f\|_{L_p(B(x_0,t))}t^{-\frac{n}{q}-1}\,dt\\ &\lesssim \Big(\sum_{|\gamma|=m-1}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\Big)\|f\|_{LM_{p,\varphi_1}^{\{x_0\}}}.\end{aligned}$$

5 Proof of Theorem 1.6

The statement is derived from the estimate (3.1). The estimation of the norm of the operator, that is, the boundedness in the non-vanishing space, immediately follows from by Corollary 1.5. So we only have to prove that

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\mathfrak{A}_{p,\varphi_1}^{\alpha,V}(f;x,r)=0 \quad\Rightarrow\quad \lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\mathfrak{A}_{q,\varphi_2}^{\alpha,V}(I_{\Omega,\alpha}^{A,m}f;x,r)=0 \tag{5.1}$$

and

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\mathfrak{A}_{1,\varphi_1}^{\alpha,V}(f;x,r)=0 \quad\Rightarrow\quad \lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\mathfrak{A}_{n/(n-\beta),\varphi_2}^{W,\alpha,V}(I_{\Omega,\alpha}^{A,m}f;x,r)=0. \tag{5.2}$$

To show that $\sup\limits_{x\in\mathbb{R}^n}\varphi_2(x,r)^{-1}r^{-n/p}\|I_{\Omega,\alpha}^{A,m}f\|_{L_q(B(x,r))}<\varepsilon$ for small r, we split the right-hand side of (3.1):

$$\Big(1+\frac{r}{\rho(x)}\Big)^\alpha\varphi_2(x,r)^{-1}r^{-n/p}\|I_{\Omega,\alpha}^{A,m}f\|_{L_q(B(x,r))}\le C[I_{\delta_0}(x,r)+J_{\delta_0}(x,r)], \tag{5.3}$$

where $\delta_0>0$ (we may take $\delta_0>1$), and

$$I_{\delta_0}(x,r):=\frac{\Big(1+\frac{r}{\rho(x)}\Big)^\alpha}{\varphi_2(x,r)}\int_r^{\delta_0}t^{-\frac{n}{q}-1}\|f\|_{L_p(B(x,t))}dt$$

and

$$J_{\delta_0}(x,r):=\frac{\Big(1+\frac{r}{\rho(x)}\Big)^\alpha}{\varphi_2(x,r)}\int_{\delta_0}^\infty t^{-\frac{n}{q}-1}\|f\|_{L_p(B(x,t))}dt$$

and it is supposed that $r<\delta_0$. We use the fact that $f\in VM_{p,\varphi_1}^{\alpha,V}(\mathbb{R}^n)$ and choose any fixed $\delta_0>0$ such that

$$\sup_{x\in\mathbb{R}^n}\Big(1+\frac{t}{\rho(x)}\Big)^\alpha\varphi_1(x,t)^{-1}t^{-n/p}\|f\|_{L_p(B(x,t))}<\frac{\varepsilon}{2CC_0},$$

where C and C_0 are constants from (1.8) and (5.3). This allows to estimate the first term uniformly in $r \in (0, \delta_0)$:

$$\sup_{x \in \mathbb{R}^n} C I_{\delta_0}(x, r) < \frac{\varepsilon}{2}, \quad 0 < r < \delta_0.$$

The estimation of the second term now my be made already by the choice of r sufficiently small. Indeed, thanks to the condition (2.5) we have

$$J_{\delta_0}(x, r) \leq c_{\sigma_0} \frac{\left(1 + \frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_1(x, r)} \, \|f\|_{VM^{\alpha,V}_{p,\varphi_1}},$$

where c_{σ_0} is the constant from (1.5). Then, by (2.5) it suffices to choose r small enough such that

$$\sup_{x \in \mathbb{R}^n} \frac{\left(1 + \frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_2(x, r)} \leq \frac{\varepsilon}{2 c_{\sigma_0} \|f\|_{VM^{\alpha,V}_{p,\varphi_1}}},$$

which completes the proof of (5.1).

The proof of (5.2) is similar to the proof of (5.1).

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